A NEW INVARIANT AND DECOMPOSITIONS OF MANIFOLDS

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Abstract. We introduce a new invariant of manifolds associated with a kind of decompositions of manifolds.

1. A problem

In order to state our problem, we prepare a definition.

Definition 1.1. Let M (resp. N) be an m-dimensional smooth connected compact manifold with boundary. Let $\partial M = \amalg X_i$ and $\partial N = \amalg Y_i$, where $\amalg$ denotes a disjoint union of manifolds. Let X_i (resp. Y_i) be connected. A *boundary union* $M \cup_\partial N$ is an m-manifold which is a union $M \cup N$ with the following properties: Let $p \in M \cap N$. Then we have:

(1) $p \in \partial M$ and $p \in \partial N$. Hence there is one and only one boundary component X_σ (resp. Y_τ) of M (resp. N) which includes the point p.
(2) X_σ is diffeomorphic to Y_τ. We identify X_σ with Y_τ when we making $M \cup_\partial N$. (Note that, in the oriented manifold case, both M and N are oriented and we let the orientations be compatible.)

Note that we do not assume how many components $M \cap N$ has.
Note that not all unions $M \cup N$ are boundary unions.

Let ρ be an integer $\geqq 2$. Suppose that a boundary union L' of ρ manifolds $L_1, ..., L_\rho$ is defined. Then a boundary union of a manifold $L_{\rho+1}$ and L' is said to be a *boundary union of* $(\rho+1)$ *manifolds* $L_1, ..., L_{\rho+1}$, be denoted by $(\cup_\partial)_{i=1}^{\rho+1} L_i$. We say that M is a *boundary union of one connected manifold* M.

We state our problem.

Problem 1.2. Let m be a nonnegative integer. Is there a finite set $\mathcal{S}$ of oriented compact connected manifolds with the following property $(\star)$?

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(⋆) An arbitrary compact connected m-manifold M with boundary is a boundary union of finite numbers of manifolds each of which is an element of $\mathcal{S}$.

We can consider the case where M is oriented. We can also do the case where the diffeomorphism type (resp. the homeomorphism type) of ∂M is restricted. We may do other cases.

It is trivial that the answer is affirmative if $m \leqq 2$. String theory uses the fact that the $m = 2$ case has the affirmative answer, discussing the world sheet (see e.g. [1, 3]).

At least, to the author, a motivation of this paper is the following: In QFT, each Feynman diagram is made by the given fundamental parts. In string theory a world sheet (so to say, 2-dimensional Feynman diagram) is decomposed into a finite number of 2-manifolds as stated above. In M-theory we may need high dimensional Feynman diagrams (see P. 607, 608 of [5] and see [1, 3]etc.). Considering high dimensional Feynman diagrams, we need to research a kind of decomposition of manifolds, e.g. in Problem 1.2.

In §6 we prove that the answer to Problem 1.2 is negative if $m \geqq 3$ and if each element of $\mathcal{S}$ has more than three connected boundary components.

2. A new invariant

We introduce a new invariant in order to discuss the $m \geqq 3$ case of Problem 1.2.

Definition 2.1. Let M be an m-dimensional smooth connected compact manifold with boundary. Take a handle decomposition $A \times [0, 1] \cup$ (handles) of M. Here, we recall the following. (See [2, 4] for handle decompositions.)

(1) The manifold A is a closed $(m-1)$-manifold $\subset \partial M$. The manifold A may be ∂M. The manifold A may not be ∂M. We may have $A = \phi$.
(2) The manifold A may not be connected.
(3) There may be no handle (then $M = A\times[0, 1]$). If handles are attached to $A\times[0, 1]$, all handles are attached to $A \times \{1\}$ not to $A \times \{0\}$.

Let $\mathcal{H}(M, A)$ denote this handle decomposition. An *ordered handle decomposition* $\mathcal{H}_O(M, A)$ consists of

(1) a handle decomposition $\mathcal{H}(M, A)$ of M, and
(2) an order of the handles h in $\mathcal{H}(M, A)$: If we give an order to the handles, let handles be called $h(\xi)$ $(\xi = 1, 2, 3, ...\delta)$.
The order satisfies the following: Let μ be a natural number $\leqq \delta$. Let $(M, A)_\mu = \cup_{j=1}^{j=\mu} h(j) \subset M$. Then $\cup_{j=1}^{j=\mu} h(j)$ is a handle decomposition of $(M, A)_\mu$. (We sometimes abbreviate $(M, A)_\mu$ to M_μ.)

(Note: if $\mu = 0$, then suppose $M_\mu = A \times [0, 1]$).

Take an ordered handle decomposition $\mathcal{H}_O(M, A)$.

Let $\partial M_\mu - A \times \{0\} = E_{\mu 1} \amalg ... \amalg E_{\mu \xi_\mu}$, where each $E_{\mu i}$ is a connected closed manifold.
Let $e_\mu(\mathcal{H}_O(M, A))$ be the maximum of $\Sigma_{*=0}^{*=m-1} \mathrm{dim} H_*(E_{\mu i}; \mathbb{R})$ for $i \in \{1, ..., \xi_\mu\}$. We sometimes abbreviate $e_\mu(\mathcal{H}_O(M, A))$ to e_μ.
Let $\nu(\mathcal{H}_O(M, A))$ be the maximum of $\{e_1, .., e_\delta\}$. Note that $\nu(\mathcal{H}_O(M, A))$ is the maximum of $\Sigma_{*=0}^{*=m-1} \mathrm{dim} H_*(E_{\mu i}; \mathbb{R})$ for all i, μ.
Let $\nu(M, A)$ be the minimum of $\nu(\mathcal{H}_O(M, A))$ for all ordered handle decompositions $\mathcal{H}_O(M, A)$.
Let $\nu(M)$ be the maximum of $\nu(M, A)$ for all A.

Note. By the definition, $\nu(M)$ is an invariant of diffeomorphism type of M. If we consider $\nu(M)$ for all smooth structures on M, we get an invariant of homeomorphism type of M.

Note. $\Sigma_{*=0}^{*=m-1} \mathrm{dim} H_*(E_{\mu i}; \mathbb{R})$ is not the Euler number of $E_{\mu i}$. Their definitions are different.

Theorem 2.2. *Let M and N be m-dimensional compact connected manifolds with boundary. Let $M \cup_\partial N$ be a boundary union of M and N. Then we have*

$$0 \leqq \nu(M \cup_\partial N) \leqq \max(\nu(M), \nu(N)).$$

By the induction, we have a corollary.

Corollary 2.3. *Let $L_1, ..., L_\rho$ be m-dimensional compact connected manifolds with boundary. Let $\cup_{\partial}{}_{i=1}^{\rho} L_i$ be a boundary union of $L_1, ..., L_\rho$. Then we have*

$$0 \leqq \nu((\cup_\partial)_{i=1}^{\rho} L_i) \leqq \max(\nu(L_1), ..., \nu(L_\rho)).$$

Claim 2.4. *The answer to the $m \geqq 3$ case of Problem 1.2 is negative if the following Problem 2.5 has the affirmative answer.*

Problem 2.5. Let m be an integer $\geqq 3$. Suppose that there is an m-dimensional compact connected manifold X. Take any natural number N. Then is there an m-dimensional compact connected manifold M such that $\partial M = \partial X$ and that

$$\nu(M) \geqq N?$$

In particular, consider the $\partial X = \phi$ case.

Note. If we do not fix the diffeomorphism type of ∂M, it is easy to prove that there are manifolds M such that $\nu(M) \geqq N$. Because: Examples are manifolds M made from one 0-handle h^0 and N' copies of h^1, where $N' \geqq N$.

3. Proof of Theorem 2.2 and Claim2.4

Proof of Theorem 2.2. By the definition of $\nu(M \cup_\partial N)$, there is an $(m-1)$-manifold P such that

$\nu(M \cup_\partial N)$=$\nu(M \cup_\partial N, P)$. —-[1]
Let $A = P \cap M$. Let $B = P \cap N$. Let $C = M \cap N$.

Suppose that an ordered handle decomposition $\mathcal{H}_O(M, A)$ gives $\nu(M, A)$. Hence $\nu(M, A) = \nu(\mathcal{H}_O(M, A))$.—-[2]
Let $\mathcal{H}_O(M, A)$ consist of ordered handles $h(1), ..., h(\alpha)$.

Suppose that an ordered handle decomposition $\mathcal{H}_O(N, B \amalg C)$ gives $\nu(N, B \amalg C)$. Hence $\nu(N, B \amalg C) = \nu(\mathcal{H}_O(N, B \amalg C))$.—–[3]
Let $\mathcal{H}_O(N, B \amalg C)$ consist of ordered handles $k(1), ..., k(\beta)$.

Let $\mathcal{H}_O(M \cup_\partial N, P)$ be an ordered handle decomposition to consist of $l(1), ..., l(\alpha + \beta)$, where the restriction of $\mathcal{H}_O(M \cup_\partial N, P)$ to $\begin{cases} (M, A) \\ (N, B \amalg C) \end{cases}$ is $\begin{cases} \mathcal{H}_O(M, A) \\ \mathcal{H}_O(N, B \amalg C). \end{cases}$ That is, we have an ordered handle decomposition

$$\begin{array}{ccccccccccccc} M \cup_\partial N = & (A \amalg B) & \cup & l(1) & \cup ... & \cup & l(\alpha) & \cup & l(\alpha+1) & \cup & ...\cup & l(\alpha+\beta) \\ & || & & || & & & || & & || & & & || \\ & P & & h(1) & ... & & h(\alpha) & & k(1) & & ... & k(\beta). \end{array}$$

Here, note that $l(i) = h(i)(i = 1, ..., \alpha)$, and that $l(j) = k(j + \alpha)(j = 1, ..., \beta)$.

Recall $e_\mu(\mathcal{H}_O(,))$ in Definition 2.1.
If $\nu(\mathcal{H}_O(M \cup_\partial N, P)) = e_\mu(\mathcal{H}_O(M \cup_\partial N, P))$ $(\alpha + 1 \leqq \mu \leqq \beta)$, then
$\nu(\mathcal{H}_O(M \cup_\partial N, P)) = \nu(\mathcal{H}_O(N, B \amalg C))$. —–[4]
Let $B = B_1 \amalg ... \amalg B_\zeta$, where B_j is a closed connected manifold.
If $\nu(\mathcal{H}_O(M \cup_\partial N, P)) = e_\mu(\mathcal{H}_O(M \cup_\partial N, P))$ $(\mu \leqq \alpha)$,
$\nu(\mathcal{H}_O(M \cup_\partial N, P)) = \max\{\nu(\mathcal{H}_O(M, A)), \Sigma_{*=0}^{*=m-1}\dim H_*(B_j; \mathbb{R})\}$ for all j.
Hence
$\nu(\mathcal{H}_O(M \cup_\partial N, P)) = \nu(\mathcal{H}_O(M, A))$ —–[5]
or
$\nu(\mathcal{H}_O(M \cup_\partial N, P)) = \Sigma_{*=0}^{*=m-1}\dim H_*(B_j; \mathbb{R})$ for an integer j. —-[6]
Let $C = C_1 \amalg ... \amalg C_\eta$, where C_i is a closed connected manifold. Note that
$e_0(\mathcal{H}_O(N, B \amalg C)) = \max\{\Sigma_{*=0}^{*=m-1}\dim H_*(B_j; \mathbb{R}), \Sigma_{*=0}^{*=m-1}\dim H_*(C_i; \mathbb{R})\}$ for all i, j.
Hence, in the [6] case,
$\nu(\mathcal{H}_O(M \cup_\partial N, P)) \leqq e_0(\mathcal{H}_O(N, B \amalg C))$.
By the definitiion of $\nu(\mathcal{H}_O(N, B \amalg C))$,
$e_0(\mathcal{H}_O(N, B \amalg C)) \leqq \nu(\mathcal{H}_O(N, B \amalg C))$.
Hence
$\nu(\mathcal{H}_O(M \cup_\partial N, P)) \leqq \nu(\mathcal{H}_O(N, B \amalg C))$. —-[7]
Note that, in the [6] case, by the above discussion,
$\nu(\mathcal{H}_O(N, B \amalg C)) = \nu(\mathcal{H}_O(M \cup_\partial N, P)) = \Sigma_{*=0}^{*=m-1}\dim H_*(B_j; \mathbb{R})$ for the integer j.
Since [4], [5], or [7] holds,

$\nu(\mathcal{H}_O(M \cup_\partial N, P)) \leqq \max\{\nu(\mathcal{H}_O(M, A)), \nu(\mathcal{H}_O(N, B \amalg C))\}$. —-[8]

By [2], [3], and [8],
$\nu(\mathcal{H}_O(M \cup_\partial N, P)) \leqq \max\{\nu(M, A), \nu(N, B \amalg C)\}$. ——[9]
By the definition of $\nu(M \cup_\partial N, P)$,
$\nu(M \cup_\partial N, P) \leqq \nu(\mathcal{H}_O(M \cup_\partial N, P))$. ——[10]
By [9] and [10],
$\nu(M \cup_\partial N, P) \leqq \max\{\nu(M, A), \nu(N, B \amalg C)\}$. ———[11]
By [1] and [11]
$\nu(M \cup_\partial N) \leqq \max\{\nu(M, A), \nu(N, B \amalg C)\}$. ——[12]
By the definition of $\nu(M)$ and $\nu(N)$, we have
$\nu(M, A) \leqq \nu(M)$, and $\nu(N, B \amalg C) \leqq \nu(N)$. ——[13]
By [12] and [13],
$\nu(M \cup_\partial N) \leqq \max\{\nu(M), \nu(N))\}$.

By the definition of $\nu(\quad)$, $0 \leqq \nu(M \cup_\partial N)$.
This completes the proof.

Proof of Claim 2.4. We suppose the following assumption and we deduce a contradiction.

Assumption: we have the affirmative answer to Problem 1.2.

By the above assumption there is a finite set $\mathcal{S} = \{S_1, ..., S_s\}$ such that an arbitrary compact m-manifold M is a boundary union of some copies of $S_i (i = 1, ..., s)$.
By Corollary 2.3, $\nu(M) \leqq \max\{\nu(S_1), ..., \nu(S_s)\}$.

If Problem 2.5 has the affirmative answer, then there is a compact m-manifold M such that $\nu(M) > \max\{\nu(S_1), ..., \nu(S_s)\}$.

We arrived at a contradiction. Hence Claim 2.4 is true.

4. Some results on our new invariant

Let $M \neq \phi$. Let $m \geq 3$. Let M be a smooth closed oriented connected m-manifold. By the definition, $\nu(M) \geqq 2$. We prove:

Theorem 4.1. *Let S^m be diffeomorphic to the standard sphere. Then $\nu(S^m) = 2$.*

Proof of Theorem 4.1. There is an ordered handle decomposition $\mathcal{H}_O = h(1) \cup h(2)$ such that $h(1) = h^0$ $h(1) = h^m$. Then $\nu(\mathcal{H}_O) = 2$. Hence $\nu(M) \leqq 2$. By the definition, $\nu(M) \geqq 2$ for any M. Hence $\nu(S^m) = 2$.
Note. Furthermore, we have the following: if M has a handle decomposition $h^0 \cup h^m$, then $\nu(\mathcal{H}) = 2$.

It is natural to ask the following: Suppose M is a closed oriented manifold. Then does $\nu(M) = 2$ implies that M is PL homeomorphic to S^m?
We have the following theorems to this question.

Theorem 4.2. *Let $m \neq 4k (m \in \mathbb{N}, k \in \mathbb{N})$. Let M be an m-dimensional connected closed oriented manifold. Suppose $\nu(M) = 2$. Then M has a handle decomposition $h^0 \cup h^m$.*

Theorem 4.3. *Let $m = 4k (k \in \mathbb{N})$. Then there is an m-dimensional connected closed oriented manifold M such that $\nu(M) = 2$ and that there is an integer $*$ such that $H_*(M; \mathbb{R})$ is NOT $H_*(S^m; \mathbb{R})$.*

Proof of Theorem 4.2. There is an ordered handle decomposition $\mathcal{H}_O$ such that $\nu(\mathcal{H}_O) = 2$.

There is an integer μ such that
$M_\mu = (h^0 \cup h^p) \amalg (l\text{-copies of } h^0)$, where $l \in \{0\} \cup \mathbb{N}$.
We suppose that $m - 1 \geqq p \geqq 1$ and we deduce a contradiction.
Let $X = h^0 \cup h^p$. Then $H_*(X; \mathbb{R}) \cong H_*(S^p; \mathbb{R})$. By Poincaré duality and the universal coefficient theorem, $H_*(X; \mathbb{R}) \cong H_{m-*}(X, \partial X; \mathbb{R})$.
Since $\nu(\mathcal{H}_O) = 2$, $H_*(\partial X; \mathbb{R}) \cong H_*(S^{m-1}; \mathbb{R})$.
Hence we have the following.

$$H_*(X; \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } * = 0, p \\ 0 & \text{if } * \neq 0, p. \end{cases}$$

$$H_*(X, \partial X; \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } * = m, m-p \\ 0 & \text{if } * \neq m, m-p. \end{cases}$$

$$H_*(\partial X; \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } * = m-1, 0 \\ 0 & \text{if } * \neq m-1, 0. \end{cases}$$

Suppose $p = 1$. Then $\partial X = S^1 \times S^{m-2}$. Then $\Sigma_{*=0}^{*=m-1} \dim H_*(\partial X; \mathbb{R}) = 4$. Hence $\nu(M) \neq 2$. We arrived at a contradiction. Hence $p \neq 1$. Hence $H_1(X; \mathbb{R}) \cong 0$.

Consider the Mayer-Vietoris exact sequence:
$H_*(\partial X; \mathbb{R}) \to H_*(X; \mathbb{R}) \to H_*(X, \partial X; \mathbb{R})$.

By this exact sequence and $H_1(X; \mathbb{R}) \cong 0$, we have $H_1(X, \partial X; \mathbb{R}) \cong 0$. Hence $m - p \neq 1$. Hence $p \neq m - 1$. Hence $p \leqq m - 2$.

By this exact sequence, $H_*(\partial X; \mathbb{R}) \cong H_*(S^{m-1}; \mathbb{R})$, and $p \leqq m-2$, we have that the homomorphism $H_p(X; \mathbb{R}) \to H_p(X, \partial X; \mathbb{R})$ in the above exact sequence is an isomorphism. Hence we have $m - p = p$. Hence $m = 2p$. Hence m is even.
Hence $m = 4k + 2$. Hence $p = 2k + 1$. Then, for any p-chain γ, the intersection product $\gamma \cdot \gamma = 0$. Hence we have that the homomorphism $H_p(X; \mathbb{R}) \to H_p(X, \partial X; \mathbb{R})$ in the above exact sequence is the zero map.
We arrived at a contradiction (in the case where $m = 4k + 2$). Hence $p = m$.

Since $h^0 \cup h^m$ is a connected closed orientable manifold, $l = 0$ and $M = h^0 \cup h^m$.

Proof of Theorem 4.3. Take the total space X of $D^{\frac{m}{2}}$-bundle over $S^{\frac{m}{2}}$ associated with the tangent bundle of $S^{\frac{m}{2}}$. Note X has a handle decomposition $h^0 \cup h^{\frac{m}{2}}$. Then ∂X is a rational homology sphere. Take the double of X, call it M. Then $H_{\frac{m}{2}}(M;\mathbb{R})$ is NOT $H_{\frac{m}{2}}(S^m;\mathbb{R})$. Note there is a handle decomposition of M which is naturally made by taking the double. It is $h^0 \cup h^{\frac{m}{2}} \cup \overline{h}^{\frac{m}{2}} \cup h^m$, where $h^{\frac{m}{2}}$ is in X and where $\overline{h}^{\frac{m}{2}}$ is NOT in X. Give an order: $h(1) = h^0, h(2) = h^{\frac{m}{2}}, h(3) = \overline{h}^{\frac{m}{2}}, h(4) = h^m$. Call this ordered handle decomposition $\mathcal{H}_O$. Then each ∂M_μ as in Definition 2.1 is a rational homology sphere (including S^{m-1}) or an empty set. Hence $\nu(\mathcal{H}_O) = 2$. Hence $\nu(M) = 2$.

Theorem 4.4. *In Theorem 2.2, there is a pair of manifolds M, N such that*

$$\nu(M \cup_\partial N) \neq \max(\nu(M), \nu(N)).$$

Proof of Theorem 4.4. Let $M \cong N$. Let M be the 2-dimensional solid torus $S^1 \times D^2$. Note ∂M is the torus T^2. Consider all $\mathcal{H}_O(S^1 \times D^2, T^2)$ and $\mathcal{H}_O(S^1 \times D^2, \phi)$. Then all these $\mathcal{H}_O$ have $\partial M_\mu = T^2$ for an integer μ. Hence $\nu(\mathcal{H}) \geqq 4$.

There is a handle decomposition $h^0 \cup h^1$ such that $\nu(\mathcal{H}_O(S^1 \times D^2, \phi)) = 4$.
There is a handle decomposition $T^2 \times [0,1] \cup h^2 \cup h^3$ such that $\nu(\mathcal{H}_O(S^1 \times D^2, T^2)) = 4$.
Hence $\nu(M) = \nu(N) = 4$.

Note that there is a boundary union $S^3 = M \cup_\partial N$. By Theorem 4.1, $\nu(S^3) = 2$.
Hence $\nu(M \cup_\partial N) = 2 < 4 = \max(\nu(M), \nu(N))$.
Hence $\nu(M \cup_\partial N) \neq \max(\nu(M), \nu(N))$.

Theorem 4.5. *Let $m \in \mathbb{N}$. Let $m \geqq 3$. Then there is an m-dimensional connected closed oriented manifold M such that $\nu(M) = 4$*

Proof of Theorem 4.5. Let $M = S^{m-1} \times S^1$. Then any handle decomposition includes one 1-handle. Hence any handle decomposition includes a ∂M_μ such that $\partial M_\mu = S^{m-2} \times S^1$. Hence $\nu(M) \geqq 4$.

There is a handle decomposition $M = h^0 \cup h^1 \cup h^{m-1} \cup h^m$. Even if we give this handle decomposition any order and get an ordered handle decomposition $\mathcal{H}_O$, we have $\nu(\mathcal{H}_O) = 4$. Hence $\nu(M) = 4$.

5. 3-manifolds and our new invariant

Suppose M is a 3-dimensional connected closed oriented manifold. Then, by the definition of $\nu(M)$ and that of the Heegaard genus of M, we have
(the Heegaard genus) $\times 2 + 2 \geqq \nu(M)$.

Theorem 5.1. *Let M be a 3-dimensional connected closed oriented manifold. If Heegaard genus is one, then $\nu(M) = 4$.*

Proof of Theorem 5.1. Since the Heggard genus is one, M is not a sphere. By Theorem 4.2, $\nu(M) \neq 2$. Since $E_{\mu i}$ in Definition 2.1 is a closed oriented surface, $\nu(M)$ is even. Hence $\nu(M) \geqq 4$.
There is a handle decomposition $\mathcal{H}$ of M such that $M = h^0 \cup h^1 \cup h^2 \cup h^3$. Even if we give $\mathcal{H}$ any order and get an ordered handle decomposition $\mathcal{H}_O$, we have $\nu(\mathcal{H}_O) = 4$. Hence $\nu(M) = 4$.

The converse of Theorem 5.1 is not true.

Theorem 5.2. *Let n be any natural number. Then there is a connected closed oriented 3-dimensional manifold M such that $\nu(M) = 4$ and that Heegaard genus of M is n.*

Proof of Theorem 5.2. Consider the connected sum of n copies of $\mathbb{R}P^3$.

It is natural to ask whether
$\nu(M) =$ (Heegaard genus) $\times 2 + 2$
is true if M is prime. It is not true in general.

Theorem 5.3. *There is a connected closed oriented 3-dimensional prime manifold M such that $\nu(M) \neq$ (Heegaard genus) $\times 2 + 2$.*

Proof of Theorem 5.3. Let M be $S^1 \times \Sigma_2$, where Σ_g is a closed oriented connected surface with genus g. Note M is prime. Then the Heegaard genus of M is five. Hence (Heegaard genus) $\times 2 + 2 = 12$.

Let N be $S^1 \times (T^2-$(an embedded open 2-disc)). We can regard M as the double of N. There is an ordered handle decomposition $\mathcal{H}_O(M)$ to consist of $h(1), ..., h(6)$ with the following properties:
(1) $h(1) = h^0$, $h(2) = h^1$, $h(3) = h^1$, $h(4) = h^1$, $h(5) = h^2$, and $h(6) = h^2$.
(2) $\partial M_1 = S^2$, $\partial M_2 = T^2$, $\partial M_3 = \Sigma_2$, $\partial M_4 = \Sigma_3$, $\partial M_5 = \Sigma_2$, and $\partial M_6 = T^2$.

Hence $\nu(N, \phi) \leqq \Sigma_{*=0}^{*=m-1} \dim H_*(\Sigma_3; \mathbb{R}) = 8$. Consider the dual handle decomposition of the above one. Hence $\nu(N, \partial N) \leqq 8$. By the definition of $\nu(N)$, $\nu(N) \leqq 8$. By Theorem 2.2, $\nu(M) \leqq \nu(N)$. Hence $\nu(M) \leqq 8$. Hence $\nu(M) \neq$ (Heegaard genus) $\times 2 + 2$.

6. The solution of a special case

We prove that the answer to Problem 1.2 is negative if $m \geqq 3$ and if each element of $\mathcal{S}$ has more than three connected boundary components.

We suppose that the following assumption is true, and deduce a contradiction.

Assumption. We have $m \geqq 3$. Each element of $\mathcal{S}$ has more than three connected boundary components. The answer to Problem 1.2 is affirmative.

Let W be an m-dimensional arbitrary compact connected manifold with boundary. Then we can divide W into pieces $W_i \in \mathcal{S}$ and can regard $W = W_1 \cup_\partial ... \cup_\partial W_w$. Consider the Meyer-Vietoris exact sequence:
$H_j(\amalg_{i,i'}\{W_i \cap W_{i'}\}; \mathbb{Q}) \to H_j(\amalg_{i=1}^{i=w} W_i; \mathbb{Q}) \to H_j(W; \mathbb{Q})$. Here, $\amalg_{i,i'}$ means the disjoint union of $W_i \cap W_{i'}$ for all (i, i'). Consider
$H_1(W; \mathbb{Q}) \to H_0(\amalg_{i,i'}\{W_i \cap W_{i'}\}; \mathbb{Q}) \to H_0(\amalg W_i; \mathbb{Q}) \to H_0(W; \mathbb{Q}) \to 0$. Note
$H_0(\amalg W_i; \mathbb{Q}) \cong \mathbb{Q}^w$ and $H_0(W; \mathbb{Q}) \cong \mathbb{Q}$.

Let $H_0(\amalg_{i,i'}\{W_i \cap W_{i'}\}; \mathbb{Q}) \cong \mathbb{Q}^\rho$. Suppose that ∂W has z components. Hence $\rho \geqq \frac{3w-z}{2}$.

We suppose $H_1(W; \mathbb{Q}) \cong \mathbb{Q}^l$. Then we have the exact sequence:
$\mathbb{Q}^l \to \mathbb{Q}^\rho \to \mathbb{Q}^w \to \mathbb{Q} \to 0$. Hence $l \geqq \rho - w + 1$. Hence $l \geqq \frac{w-z+2}{2}$. Hence $(2l+z-2) \geqq w$.

We define an invariant. Let X be a compact manifold. Take a handle decomposition of X. Consider the numbers of handles in the handle decompositions. Let $h(X)$ be the minimum of such the numbers.

Suppose that $\mathcal{S}$ is a finite set $\{M_1, ..., M_\mu\}$. Suppose that M' is one of the manifolds M_i and that $h(M') \geqq h(M_i)$ for any i. Then we have $w \times h(M') \geqq h(W)$. Hence $(2l + z - 2) \times h(M') \geqq h(W)$. Note that the left side is constant.

For any natural number N, there are countably infinitely many compact oriented connected m-manifolds W' with boundaries such that $\partial W' = \partial W$ that
$H_1(W; \mathbb{Q}) \cong \mathbb{Q}^l$, and that $h(W) \geqq N$. Because: There is an n-dimensional manifold P such that $H_1(P; \mathbb{Q}) \cong \mathbb{Q}^l$. There is an n-dimensional rational homology sphere Q which is not an integral homology sphere. Make a connected sum which is made from one copy of P and q copies of Q ($q \in \mathbb{N} \cup \{0\}$).

We arrived at a contradiction. This completes the proof.

Furthermore, [**?**] pointed out the following.
(1) There is a piece of n-dimensional Feynman diagram with three outlines with the following properties. Two copies of the piece is made into countably infinitely many kinds of tree diagrams with four outlines.
(The idea of the proof: Let the piece be {(the solid torus)$-$two open 3-balls}. Use the fact that all lens spaces, S^3, and $S^1 \times S^2$ are made from two solid torus.)
(2) There is an infnite set $\mathcal{S}$ with the following properties.
(i) All m-dimensional Fyenman diagrams (compact manifolds) are boundary sums of finite elements of $\mathcal{S}$.
(ii) Each element of $\mathcal{S}$ is what is made by attaching a handle to (($m-1$)-dimensional closed manifold)$\times[0, 1]$. Note it has one, two or three boundary components. (The idea of the proof: Use handle decompositions.)

7. Discussion

Take a group $G = \{g_1, ..., g_N|$
$g_1 \cdot g_2 ... \cdot g_{N-1} \cdot g_N \cdot g_2^{-1} ... \cdot g_{N-1}^{-1} = 1,$
$g_2 \cdot g_3 ... \cdot g_N \cdot g_1 \cdot g_3^{-1} ... \cdot g_N^{-1} \cdot g_1^{-1} = 1, ...,$
$g_N \cdot g_1 ... \cdot g_{N-2} \cdot g_{N-1} \cdot g_1^{-1} ... \cdot g_{N-1}^{-1} = 1.\}$

In the $m \geqq 4$ case, we can make a compact connected oriented manifold Z such that
(1) $\pi_1(Z) = G$.
(2) Z is made of one 0-handle, N copies of 1-handles, and N copies of 2-handles.
Take the double of Z. Call it W. Note $\pi_1(W) = G$.
Thus we submit the following problem.

Problem 7.1. Do you prove $\nu(W) \geqq N$?

If the answer to Problem 7.1 is affirmative, then the answer to Problem 2.5 is affirmative (in the closed manifolds case, which would be extended in all cases).
By using a manifold whose fundamental group is so complicated as above, we may solve Problem 1.2, 2.5.

Use Z_p coefficient homology groups instead in the definition of ν. Use the order of Tor$H_*(\quad; \mathbb{Z})$ instead in the definition of ν. Can we solve Problem 1.2?

Calculate ν of the knot complement. (In particular, in the case of 1-dimensional prime knots. In this case, what kind of connection with the Heegaard genus?)

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